Technical Comments

Solving Problems of Elasto-Plastic Flow

J. L. SWEDLOW*
Carnegie-Mellon University, Pittsburgh, Pa.

IN Ref. 1, Akyuz and Merwin document a solution technique for a class of problems of increasing engineering interest. It would appear from the presentation of their work that two comments are in order. First, the authors have failed to recognize the essential mathematical nature of their problem in that they refer to it as nonlinear. Second, many of the procedural aspects of their work have been predated.

That the problem is not nonlinear may be deduced by observing carefully the structure of the governing equations. In addition, the basic relations imply the same information. The authors recognize the linear relation between strain and stress increments which, in compact notation, takes the form

$$d\epsilon_{ij} = {}_{ijkl}d\sigma_{kl}, \quad \mathfrak{D}i,j,k,l = 1,2,3 \tag{1}$$

In (1), $d\epsilon_{ij}$ and $d\sigma_{kl}$ are these increments; $\mathfrak{D}_{ijkl}$ is a compliance that depends on the instantaneous values of the stresses and the stress-strain relation the material obeys.

If (1) is combined with a) equilibrium equations in terms of the stress increments and b) strain-displacement equations also in terms of increments, a complete problem in terms of incremental quantities may be defined. This problem has several interesting features. It requires both boundary and initial conditions. It is neither linear nor nonlinear (in terms of incremental quantities), but it is quasi-linear; that is, the dependent variables occur in a manner such that their highest derivatives are linear. Thus, although the physical event remains nonlinear, the mathematical problem is quasi-linear.

If, in addition, the stress-strain curve is presumed to be monotonic, with a continously turning tangent—as Akyuz and Merwin have done—an additional feature accrues. The governing equations may be shown to be elliptic. In this case, there are no slip lines or discontinuities of any sort in the field. This feature has enormous implications for numerical analysis of which Akyuz and Merwin take advantage. During the initial load increment, when the response is wholly elastic, the behavior is, of course, elliptic. The same behavior is guaranteed for all subsequent load increments so that they never need be concerned with the transition to slip line generation. A more detailed statement and derivation of these characteristics of the problem appear elsewhere.[2]

That some of the procedures have been worked out previously may be seen by consulting the literature; see, e.g., Refs. 3–8.

References

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[2] Swedlow, J. L., "Character of the Equations of Elasto-Plastic Flow in Three Independent Variables," *International Journal of Non-Linear Mechanics*, Vol. 3, No. 3, Sept. 1968, pp. 325–336.

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[4] Marçal, P. V., "A Comparative Study of Numerical Methods of Elastic-Plastic Analysis," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 157–158.

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[7] Swedlow, J. L., Williams, M. L., and Yang, W. H., "Elasto-Plastic Stresses and Strains in Cracked Plates," *Proceedings of the First International Conference on Fracture*, Vol. 1, 1965, pp. 259–282.

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* Assistant Professor, Department of Mechanical Engineering. Member AIAA.

Reply by Author to J. L. Swedlow

FEVZICAN A. AKYUZ*
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

THE following two paragraphs are written in the hope of clarifying the confusion that arose in the mind of the author of Ref. 1 and corresponding to the two comments mentioned therein.

1) The term "nonlinear" in the title of Ref. 2 refers (as is clear from the reading and understanding of Ref. 2) to the facts that the geometry of the boundary and the prevailing conditions therein are nonlinear functions of the unknown variables and that the stiffness due to the initial stresses and incremental rotations is taken into account. Since these are prominent aspects of the paper, the authors deliberately included the term nonlinear in the title. In addition to these obvious reasons for the term nonlinear one can read on page 1827 of Ref. 2, following Eq. (19), that "Equations (16) and (17) constitute a set of nonlinear equations for the solution of $d\sigma_{zz}$," which means that in the plane strain case Eq. (1) of Ref. 1 becomes mathematically nonlinear, exactly in the sense the author in Ref. 1 would like to see it.

2) At the bottom of page 1825 of Ref. 2, a note "Presented as Paper 67-144 at the AIAA 5th Aerospace Sciences Meeting, New York, January 23–26, 1967" eliminates the possibilities of Ref. 2 being predated by the list of Refs. 1–4 and 8 cited in Ref. 1. Actually, as far as procedural aspects of these works are concerned, Ref. 3 presents clearly the derivation of Prandtl-Reuss equations for both perfectly plastic material and for material with strain hardening. Reference 4 treats the incremental load technique, Ref. 5 treats the application of finite element and direct stiffness method with incremental load technique, and Ref. 6 indicates the application of Prandtl-Reuss equations to the finite-element and force method. The authors regret having missed any additional information in Refs. 5–7 of Ref. 1 during the preparation of their paper.

As explained in the foregoing comment 1) the intrinsic nature of the problem treated in Ref. 2 is totally different from the ones treated in Refs. 1–8 of Ref. 1 and Refs. 4–6 of this article. Furthermore, the complexity of the problem in Ref. 2, in contrast to the simplicity by which the problems could be treated in the above references, required a strong positive definite behavior of the stiffness matrix at each in-

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* Senior Research Engineer.